On the Asymptotic Behavior of Discrete ... Osama Abd-Alwahab Al-Khatatneh Mu'tah University, 2016

Mu'tah University
College of Graduate Studies

On the Asymptotic Behavior of Discrete Dynamical Systems Using Bi-shadowing Properties

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DEDICATION

It is my pleasure to dedicate this work to my parents, brother, sisters, friends and to everyone who has always helped and support me.

Osama A. Al-khatatneh

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I would like to express my Gratitude and special thank to my supervisor Dr. Anwar Al-Badarneh for his supervision, without his guidance and encouragement this thesis could never have been completed.

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Osama A. Al-khatatneh

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ABSTRACT

On the Asymptotic Behavior of Discrete Dynamical Systems Using Bi-shadowing Properties

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Mu'tah University, 2016

In this thesis, we study the asymptotic properties of discrete dynamical systems given by a continuous mappings on a metric space.

In particular, we show that if a system has the shadowing property, then any system conjugated with it has the same property. Also, we investigate the relationship of shadowing property between the product system and its subsystems. We generalize these results in the context of bi-shadowing property for dynamical system under certain conditions.

الملخص

السلوك التقاربي للأنظمة الديناميكية المتقطعة
باستخدام خصائص الظل الثنائية

أسامة عبد الوهاب الختاتنة

جامعة مؤتة، 2016

في هذه الأطروحة، نقوم بدراسة الخصائص التقاربية للأنظمة الديناميكية المتقطعة و المتولدة من اقترانات متصلة ومعرفة على نظام متري.

بشكل خاص، أثبتنا أنه إذا كان النظام يحقق خاصية الظل، فأن أي نظام مترافق له يحقق نفس الخاصية ونبحث أيضا علاقة خاصية الظل بين النظام الضربي و الأنظمة الجزئية منه. كما قمنا بتعميم هذه النتائج للخاصية ثنائية الظل للأنظمة الديناميكية تحت شروط معينة.

CHAPTER ONE
Introduction

1.1 Statement of the problem

As a simple mathematical model that describes the number of bacteria in a population is the function $f(x) = 2x$, where x denotes the number of bacteria in the population and $f(x)$ denotes the number of bacteria one day later. Clearly the population doubles every day. If we assume that a population has started with 10^6 bacteria then according to the rule above, the population has $f(10^6) = 2 \times 10^6$ bacteria after one day and $f\big(f(10^6)\big) = f(2 \times 10^6) = 4 \times 10^6$ after two days. Note that after three days, the population has $f^3(10^6) = 8 \times 10^6$ bacteria, and so on.

This simple model contains about states that represent the number of bacteria in the population, in which it is changing with time under the rule described above, that is

$$x_n = f(x_{n-1}) = 2x_{n-1}.$$

Here *n* represents the time (day in our example) and x_n represents the number of bacteria at time *n*.

This model describes a dynamical system, which consists of a set of states and a rule that describes the current state in terms of previous one.

If the rule is applied at discrete times, then the system is called discrete time dynamical system. The counterpart of this system is the so called continuous time dynamical system which can be obtained by taking the limit of discrete time system over a small time. Generally, iterations of continuous function generates a discrete time system, while, a differential equation generates a continuous time system in terms of the flow map defining the solution of the differential equation.

In this thesis we deal with discrete-time dynamical system which is very interesting and exciting topic in mathematics which, in many cases, has been discovered only in the last few decades. The theory of dynamical systems is a branch of mathematics that attempts to understand processes in motion. Such processes occur in all branches of science. For example, the motion of the stars and the galaxies in the heavens is a dynamical system, one that has been studied for centuries by thousands of mathematicians and scientists. The stock market is another system that changes in time, as is the world weather. The changes of chemicals undergo, the rise and fall of populations, and the motion of a simple pendulum are classical examples of dynamical systems in chemistry, biology, and physics, respectively.

There are several references investigating and studying dynamical systems, for example (Devaney, 1989), (Irwin, 1980), (Nitecki, 1971), and (Palis and de Melo, 1982).

One of the most rapidly developing components of the global theory of dynamical systems is the theory of pseudo-orbit tracing property, or shadowing property. Pseudo-(or approximate-) trajectories arise due to the presence of round-off error, method error, and other error in computer simulation of dynamical systems. Consequently, in numerical modeling we can compute a trajectory that is coming very close to an exact solution and the resulting approximate solution will be pseudo-trajectory. Section (2.3) in this thesis presents mathematical definitions and some important theorems regarding shadowing property of dynamical systems.

Another type of shadowing properties (inverse shadowing properties) is related to the following problem: given a family of mappings that approximate the defined mapping for the dynamical system considered, can we find for a chosen exact trajectory, a close pseudo-trajectory generated by the given family? such a property were considered by many authors, for example Corless and Pilyugin (1995) and Palmer (2000).

The main subject of this thesis is to study the bi-shadowing property, which is more general property than shadowing and inverse shadowing. It was introduced by Diamond *et. al*. (1995) see also Diamond *et. al*. (2012). Bi-shadowing was considered for set-valued systems with an application to iterated function system by Al-Badarneh (2014) and for infinite dimensional dynamical systems by Al-Nayef (1997).

1.2 Motivation

This work is motivated by the following reasons:

1) As it is well-known that we can solve explicitly only very few differential equations and thus qualitative properties of solutions should be considered. So, long-term behavior of solutions of ordinary differential equations should be studied using dynamical systems techniques.

2) In the last few decades, the interest in the study of dynamical systems has increased rapidly. Partly by the discovery of the chaotic properties of dynamical systems and by the ongoing research which has a rich and many new properties in the behavior of the systems.

3) Bi-shadowing is an extension to the concept of shadowing and it is usually used in the context of comparing computed trajectories with the true trajectories of the dynamical system. The motivation to study systems

with these properties is that numerical simulation of dynamical systems always producing pseudo trajectories. Thus, systems with the shadowing property are precisely the ones in which numerical simulation does not introduce unexpected behavior, in the sense that simulated trajectories actually follow real trajectories. It is clear, the study of such properties is very important for the theory of perturbations of dynamical systems.

1.3 Synopsis of the Thesis

Following this brief introductory chapter, in Chapter 2 we give some definitions and preliminaries needed throughout this thesis. We give the notations of hyperbolicity and pseudo orbit tracing property or shadowing property and study how it is related to hyperbolicity. More precisely we prove that the hyperbolic linear homeomorphism of a Banach space has the shadowing property. Also, we prove that if the system (X, f) is topologically conjugate to (Y, g), then f has the shadowing property if and only if g has the same property. Moreover, we proved more results regarding the concept of shadowing property.

In Section (3.1) of Chapter 3, we discuss asymptotic behavior of dynamical systems generated by continuous almost contractive single-valued mapping defined in a metric space and proved that such maps have the bi-shadowing property. In Section (3.2), we explore more results on bi-shadowing and introduce a definition of new property of bi-shadowing. Also, we discuss the relationship of bi-shadowing property between the product system and its subsystems.

Finally, we prove that the bi-shadowing property is invariant under topological conjugacy.

1.4 Contributions

The following Theorems are proved in this thesis as an original results:
Theorem 3.2.1, 3.2.2, 3.2.3 and 3.2.4 .

CHAPTER TWO
Basic Properties of Dynamical Systems

In this introductory chapter, we give some basic concepts and definitions that we need throughout the thesis. We will also discuss important properties of dynamical systems, such as, hyperbolicity and shadowing.

A metric space is a pair (X, d) consists of a non empty set X and a function $d: X \times X \to [0,\infty)$ such that:

1) For all $x, y \epsilon X$, $d(x,y) \geq 0$ with equality if and only if $x = y$.

2) For all $x, y \epsilon X$, $d(x, y) = d(y, x)$.

3) For all $x, y, z \epsilon X$

$$d(x,y) \leq d(x, z) + d(z, y).$$

A metric space (X, d) is said to be complete if every Cauchy sequence in X converge to a point in X.

A homeomorphism is a continuous bijective mapping $f: X \to Y$ whose inverse is continuous. If both f and f^{-1} are differentiable, then a homeomorphism is said to be diffeomorphism.

2.1 Dynamical systems

The modern theory of dynamical systems has relatively short history. It begins with the work of Hennri Poincaré, who revolutionized the study of nonlinear differential equations by introducing the qualitative techniques. Therefore, geometrical and topological properties are used to obtain global properties of solutions of differential equations rather than using the analytic method of obtaining explicit solutions of the differential equations. Poincaré point of view was adopted and furthered by Birkhoff in the first part of the twentieth century. Birkhoff realized the importance of the study of mappings arising from differential equations and the relation between the behavior of the dynamical systems generated by these mappings and the original systems.

A dynamical system is usually classified as either a continuous-time dynamical system or a discrete-time dynamical system. Recently, however, discrete dynamical systems have received considerable attention. This does not mean that continuous systems declined in importance. Rather, mathematicians study discrete systems with an eye toward applying their results to the more continuous case.

A continuous-time dynamical system on a topological space X is a continuous map $\phi: R \times X \to X$ such that, for all $x \in X$ and, for all s, t $\in R$

$$\phi(s + t, x) = \phi(s, \phi(t, x)), \text{ and}$$

$$\phi(0, x) = x.$$

Example 2.1.1 (Irwin, 1980): For any X, the trivial dynamical system is defined by $\phi(t, x) = x$.

Example 2.1.2 (Irwin, 1980): For $X = R$, $\phi(t, x) = e^t x$ defines a dynamical system on X.

As an interesting example of a continuous-time dynamical system, let U be an open subset of R^n and $f: U \to R^n$ be a differentiable function. Consider the differential equation

$$x' = f(x), \quad x \in U \tag{2.1}$$

For $x_0 \in U$ and $t \in I(x_0)$, let $\phi(t, x_0)$ be the unique solution of the equation (2.1) with the initial condition $x(0) = x_0$, where $I(x_0)$ is the maximal interval for which the solution $\phi(t, x_0)$ exist and unique with $x(0) = x_0$. The function $\phi(t, x)$, $x \in U$, $t \in I(x)$ satisfies the conditions mentioned above and therefore is a continuous-time dynamical system. It is called a flow of the differential equation (2.1). More examples of continuous time dynamical systems can be obtained by considering the flow of functional differential equations and partial differential equations. In these two examples, the state space is infinite dimensional, and thus the dynamical systems are considered as infinite dimensional.

The space X in the preceding definition is called the phase space of ϕ, which could be a Banach space, or even a metric space. Let ϕ be a dynamical system of X. Given $t \in R$, we define the map $\phi^t: X \to X$ by $\phi^t(x) = \phi(t, x)$, which is called the time-t map of ϕ. If the set of real numbers R is replaced by the set of integers Z, then ϕ is called a discrete-time dynamical system, which is completely determined by ϕ^1, which is a homeomorphism on X. The main goal of the study of dynamical systems is to understand the long term behavior of states in a system for which there is a deterministic rule for how a state evolves. In the recent years, the modern theory of dynamical systems has focused on a discrete-time systems generated by a diffeomorphism f representing the time-1 map generated by solution of ordinary differential equation.

For a continuous mapping $f: X \to X$ which is identified with a discrete-time system generated by iterations of f such that $x_{i+1} = f(x_i)$ so

$$x_i = f^i(x_0)$$

where $f^i = f \circ \ldots \circ f$ is the composition of f with itself i times, with $f^i = (f^{-1})^{|i|}$ if $i < 0$ and f is invertible.

The forward trajectory from a point $x_0 \epsilon X$ is the set $\{f^i(x_0) : i \geq 0\}$. If f is invertible, then the backward trajectory $\{f^i(x_0) : i \leq 0\}$. The whole trajectory is the set $\{f^i(x_0) : i \epsilon Z\}$. If f is not invertible then we sometimes make choices and construct $x_{-1}, x_{-2}, \ldots$ where $f(x_{i-1}) = x_i, i < 0$.

Generally, a trajectory of the discrete-time dynamical system generated by a mapping f is a finite or infinite sequence $\{x_i\} \subset X$ satisfying $x_{i+1} = f(x_i)$ for $i = -N_-, \ldots, -1, 0, 1, \ldots, N_+$ where $0 \leq N_-, N_+ \leq \infty$.

A point x_0 is called a periodic point of period n provided $f^n(x_0) = x_0$ and $f^j(x_0) \neq x_0$ for $0 < j < n$. (Note that n is the least period.) If x_0 has period one then it is called a fixed point. If x_0 is a point of period n, then the forward trajectory of x_0, is called a periodic trajectory. Finally , a point x_0 is eventually periodic of period n provided that there exists an $m > 0$ such that $f^{m+n}(x_0) = f^m(x_0)$, so $f^{j+n}(x_0) = f^j(x_0)$ for $j \geq m$, and $f^m(x_0)$ is a periodic point.

2.2 Hyperbolic dynamical systems

The concept of hyperbolicity played an important role in the development of the theory of dynamical systems. Hyperbolic sets for diffeomorphisms was studied by Anosov (1967), Smale (1967), and Hirsch, Pugh and Shub (1977). Hyperbolicity is a characteristic property for complicated behavior of both systems discrete-time and continuous-time as well.

The concept of a hyperbolic set is a natural generalization of hyperbolicity of a fixed point to more general invariant sets. A set $S \subset X$ is said to be positively invariant if $f(S) \subset S$, negatively invariant if $f^{-1}(S) \subset S$ and invariant if $f(S) = S$.

Let $(E, \|.\|)$ denotes a Banach space over R. A fixed point x of a diffeomorphism mapping $f: X \to E$ ($X \subset E$ open) is said to be hyperbolic if the derivative map $Df(x)$ is hyperbolic, i.e, has no spectral values in the unit circle of the complex plane. For non-invertible continuous linear mappings we have the following definition taken from (Lani-Wayda, 1995):

Definition 2.2.1: A continuous linear mapping $f: E \to E$ is called hyperbolic when it satisfies the following two equivalent conditions:

i) The spectrum $\sigma(f)$ is disjoint from the unit circle.

ii) There exist positively invariant (with respect to f) closed subspaces E^s and E^u, that is with $f(E^s) \subset E^s$and $f(E^u) \subset E^u$, such that

$$E = E^s \oplus E^u$$

and constants k_s , $k_u > 0$, q_s ,$q_u \in (0,1)$ such that the map $f\,|_{E^u}$ is a homeomorphism on E^u and for each non-negative integer m

$$\|(\,f|_{E^s})^m\| \le k_s\; q_s^m \quad \text{and} \quad \|(\,f|_{E^u})^{-m}\| \le k_u\, q_u^m.$$

The proof that i) and ii) are equivalent can be found in (Lani-Wayda, 1995).

The subspaces $E^s$and E^u are called the stable and unstable subspaces, respectively, for the hyperbolic linear mapping f and can be characterized by:

$$E^s = \{x \in E : \{f^n x\}_{n\ge 0} \text{ is a bounded sequence}\}$$

and

$E^u = \{x \in E : \exists$ bounded sequence $\{x_n\}_{n\le 0}$ with $f(x_{n-1}) = x_n$ for $n \le 0$ and $x_0 = x\}$. In addition, the corresponding projection defined by $P^s : E \to E^s$ and $P^u : E \to E^u$, can be given by:

$$\mathrm{P^s} = \frac{-1}{2\pi i}\int_{|z|=1} (\,f - zI\,)^{-1}\, dz,$$

where I denotes the identity mapping on E and $P^u = I - P^s$. This representation of $\mathrm{P^s}$ is given in (Riesz and Nagy, 1987).

Here, we present another definition of a hyperbolic set K , which was introduced by Steinlein and Walther in (1989). Let $f : X \to E$ be a mapping, where X is an open subset of E and $\mathrm{L}(E)$ denotes the space of continuous linear maps $E \to E$.

Definition 2.2.2: A hyperbolic set for f is a positively invariant set $K \subset E$ together with a bounded, uniformly continuous mapping $P^u : K \to \mathrm{L}(E)$, $x \mapsto P_x^u$ with constant $k \ge 1$, $q \in (0,1)$ such that $E_x^u := P_x^u\, E$, $P_x^s := I - P_x^u$ and $E_x^s := P_x^s\, E$ $(x \in K)$, P^u is a projection satisfying the following properties:

H1) $Df_x(E_x^s) \subset E_{f(x)}^s$

H2) $E = E_{f(x)}^s + Df_x(E_x^u)$

H3) the inequalities

$$\left\|Df_x^n|_{E_x^s}\right\|_E \le kq^n \quad \text{and} \quad \left\|P_{f^n(x)}^u Df_x^n|_{E_x^u}\right\|_E \ge k^{-1}\, q^{-n} \text{ hold.}$$

2.3 Shadowing Properties

The numerical simulation of discrete-time dynamical systems using arithmetic algorithms generates round-off and truncation errors. This affects naturally the value of each individual iteration of the map that generates the system. Therefore, it is crucial to find a convenient way to confirm that numerical calculations, and hence the approximated system, reflect the behavior of the original system. This is usually done in the sense whether to any given approximated trajectory there always exist a true trajectory of the system nearby. This property is known as the shadowing property.

The idea of shadowing for hyperbolic dynamical systems was originally given by Anosov (1967), see also Bowen (1970). Since then, shadowing plays an important role in the investigation of the theory of dynamical systems for the continuous time systems and later for discrete time systems. Sinai (1972) stated the Shadowing Lemma for Anosov diffeomorphisms, the proof being a variation ideas from Anosov (1967). The first formal statement of the shadowing lemma for more general diffeomorphisms is given in (Bowen, 1975). However, Bowen states that the proof is already contained in the proofs of results in (Bowen, 1970) concerning the specification property. Conley (1980) also used shadowing lemma to show that if the chain recurrent set of a diffeomorphism is hyperbolic, then the periodic points are dense in the chain recurrent set. Walter (1978) and Lanford (1983) used it also to prove topological conjugacy results for perturbations of diffeomorphisms with hyperbolic set. For flow different versions of the shadowing lemma have been proved in (Frank and Selgrade , 1977), (Nadzieja, 1991), (Katok and Hasselblatt, 1995), (Coomes, Kocak and Palmer, 1995) and (Pilyugin, 1997).

The most common version of the shadowing lemma is a basic result in the theory of dynamical systems and it says that a dynamical system defined by a diffeomorphism on a compact space has the shadowing property on its hyperbolic set. In this section we give a proof due to (Ombach, 1993), see also (Katok, 1975), of the shadowing lemma for discrete systems in the simplest but nontrivial situation. Namely, we show that any linear homeomorphism of a Banach space, which is hyperbolic has also the shadowing property. The proof can be extended for noninvertible maps and for continuous-time systems.

Let (X, d) be a metric space, and let $f: X \to X$ be a continuous map. Recall that an orbit (trajectory) of f is a sequence $\{x_i\}_{i \epsilon Z}$ satisfying

$$x_{i+1} = f(x_i), \tag{2.2}$$

for all $i \in Z$. If f is a homeomorphism and $x \epsilon X$, the orbit of the point x is the sequence $\{f^i(x)\}_{i \epsilon Z}$. If a sequence $\{ x_i\}_{i \epsilon Z}$ satisfies the above equality up to some perturbation we say it is a pseudo-orbit. More precisely, we say that a sequence $\{ x_i\}_{i \epsilon Z}$ is a δ-pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$, where $\delta > 0$. Now we say that a map f has the pseudo-orbit tracing property, or the shadowing property if for every $\varepsilon > 0$ there exist $\delta > 0$ such that any δ-pseudo-orbit $\{x_i\}_{i\epsilon Z}$ is ε-traced or ε-shadowed by some orbit $\{y_i\}_{i\epsilon Z}$, i.e $d(x_i,y_i) < \varepsilon$, for all $i \in Z$.

Shadowing Lemma(Ombach, 1993): A hyperbolic linear homeomorphism of a Banach space has the shadowing property.

Let $(X, \|.\|_X)$ be a Banach space , and $f: X \to X$ a linear continuous map. If f is hyperbolic, that is the spectrum of f is disjoint from the unit circle in the complex plane. It follows from (Irwin, 1980) that there exist a decomposition $X = X_1 \oplus X_2$ and $f = f_1 \oplus f_2$, where $f_i: X_i \to X_i$ are linear continuous , i =1, 2, and there exist norms $\|.\|_i$ on X_i, i =1, 2 such that:

(i) $\|f_1\|_1 < 1$, f_2 is a homeomorphism and $\|f_2{}^{-1}\|_2 < 1$.

(ii) the norm $\|.\|$ defined by $\|x\| = \|x_1\|_1 + \|x_2\|_2$ is equivalent to the original norm of X . If f is a homeomorphism then so is f_1.

Actually, this is the definition of hyperbolicity that Ombach (1993) used to prove the shadowing lemma in which the following two preliminary lemmas are given without proofs. Although the proofs are straightforward, we shall provide a proof for them by modifying a proof for similar results in nonautonomous systems given in (Thkkar and Das, 2014 b).

Lemma 2.3.1 (Ombach, 1993): Let (X_j, d_j), j =1, 2, be metric spaces and $f_1: X_1 \to X_1$ and $f_2: X_2 \to X_2$ be two maps. Let $X = X_1 \times X_2$ be equipped with a metric generating the product topology. Let $f = f_1 \times f_2$ be a map on X defined by $f(x_1, x_2) = \big(f_1(x_1), f_2(x_2)\big)$. Then f has the shadowing property if and only if f_1 and f_2 do.

Proof: Define metric d on the product space $X_1 \times X_2$ by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\},$$

for all $(x_1, y_1), (x_2, y_2) \epsilon X_1 \times X_2$.

Given $\varepsilon > 0$. Since f has the shadowing property, then there exist $\delta > 0$ such that for every δ-pseudo trajectory of f can be ε-traced by a true trajectory of f. Let $\{x_i\}_{i\epsilon Z}$ and $\{y_i\}_{i\epsilon Z}$ are δ-pseudo trajectories of $f_1$and f_2 respectively. Hence $d_1(f_1(x_i), x_{i+1}) < \delta$ and $d_2(f_2(y_i), y_{i+1}) < \delta$ which implies by the definition of d that

$$d((f_1 \times f_2)(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta, \quad \text{for } i \in Z.$$

Thus, $\{(x_i, y_i)\}_{i\epsilon Z}$ is a δ-pseudo trajectory of f. But f has the shadowing property, so there exist a true trajectory $\{(z_i, w_i)\}_{i\epsilon Z}$ of f such that

$$d\big((z_i, w_i), (x_i, y_i)\big) < \varepsilon, \quad \text{for } i \in Z.$$

which implies

$$d_1(z_i, x_i) < \varepsilon \text{ and } d_2(w_i, y_i) < \varepsilon \qquad \text{for } i \in Z.$$

for all $i \in Z$. That is both f_1 and f_2 have the shadowing property.

Conversely, Given $\varepsilon > 0$. Then since f_1 and f_2 have the shadowing property, then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that every δ_1-pseudo trajectory of f_1 and δ_2-pseudo trajectory of f_2 can be ε-traced by a true trajectory of f_1 and f_2 respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and $\{(x_i, y_i)\}_{i\epsilon Z}$ be δ-pseudo trajectory of f. Hence

$$d((f_1 \times f_2)(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta, \qquad \text{for } i \in Z,$$

which implies,

$$d_1(f_1(x_i), x_{i+1}) < \delta < \delta_1 \text{ and } d_2(f_2(y_i), y_{i+1}) < \delta < \delta_2.$$

Thus, there exist a true trajectories $\{z_i\}_{i\epsilon Z}$ of f_1 and $\{w_i\}_{i\epsilon Z}$ of f_2 such that

$$d_1(z_i, x_i) < \varepsilon \text{ and } d_2(w_i, y_i) < \varepsilon, \qquad \text{for } i \in Z.$$

Hence by the definition of d implies

$$d\big((z_i, w_i), (x_i, y_i)\big) < \varepsilon, \qquad \text{for } i \in Z.$$

Thus f has the shadowing property. □

A sketch of the proof of the following lemma is given in (Thkkar and Das, 2014 a) for the nonautonomous case, here we provide a detailed proof.

Lemma 2.3.2 (Ombach, 1993): Let f be a homeomorphism of a metric space (X, d) such that f and its inverse f^{-1} are uniformly continuous. Then f has the shadowing property if and only if f^{-1}does.

proof: Given $\varepsilon > 0$. If f has the shadowing property, then there is $\delta > 0$, so that for each $\{x_i\}_{i\epsilon Z} \subset X$ with

$$d(f(x_i), x_{i+1}) < \delta \quad \text{for } i \in Z, \tag{2.3}$$

there exists a point $x \in X$ such that

$$d(f^i(x), x_i) < \varepsilon \quad \text{for } i \in Z.$$

By uniform continuity we choose $\delta_1 > 0$ so that $d(x, y) < \delta_1$ implies $d(f(x), f(y)) < \delta$. Let $\{y_i\}_{i\epsilon Z} \subset X$ be a δ_1-pseudo trajectory of f^{-1}, that is satisfy the inequality

$$d(f^{-1}(y_i), y_{i+1}) < \delta_1 \quad \text{for } i \in Z.$$

Then

$$d(y_i, f(y_{i+1})) = d\left(f\left(f^{-1}(y_i)\right), f(y_{i+1})\right) < \delta \qquad \text{for } i \in Z.$$

Thus, the sequence $\{x_i : x_i = y_{-i}\}$ is a δ-pseudo trajectory of f and satisfies the relation (2.3). Since f has the shadowing property, then there exists a point $x \in X$ such that

$$d(f^{-i}(x), y_i) = d(f^{-i}(x), x_{-i}) = d(f^{i}(x), x_i) < \varepsilon \qquad \text{for } i \in Z.$$

For the proof of the other side it is enough to replace f^{-1} by f. □

Proposition 2.3.1 (Ombach, 1993) Let X be a complete metric space and $f: X \to X$ a contraction, i.e there exist a constant $0 \leq \lambda < 1$ such that

$$d\big(f(x), f(y)\big) \leq \lambda\, d(x, y)$$

for all $x, y \in X$. Then f has the shadowing property.

Proof: Fix $\varepsilon > 0$ and define $\delta = (1 - \lambda)\varepsilon$. Let **x**= $\{x_i\}_{i\epsilon Z}$ be a δ-pseudo trajectory. Define a metric space **E** by

$$\mathbf{E} = \{\mathbf{y} : \mathbf{y} = \{y_i\}_{i\epsilon Z}, d(x_i, y_i) \leq \varepsilon\}$$

with metric

$$D(\mathbf{y}, \mathbf{z}) = \sup\{d(y_i, z_i) : i \,\epsilon\, Z\}.$$

It is easy to see that $(\mathbf{E}, D)$ is complete. Consider a map **F** defined for $\mathbf{y} \,\epsilon\, \mathbf{E}$ by

$$\mathbf{F}(\mathbf{y})_i = f(y_{i-1}) \qquad \text{for } i \,\epsilon\, Z.$$

For all $i \,\epsilon\, Z$ we have

$$\begin{aligned} d(f(y_{i-1}), x_i) &\leq d(f(y_{i-1}), f(x_{i-1})) + d(f(x_{i-1}), x_i) \\ &\leq \lambda\, d(y_{i-1}, x_{i-1}) + \delta \\ &\leq \lambda\, \varepsilon + (1 - \lambda)\varepsilon = \varepsilon \end{aligned}$$

which means that $\mathbf{F}(\mathbf{E}) \subset \mathbf{E}$. Also, we can easily see that **F** is contraction with contraction constant λ, since

$$\begin{aligned} D\big(\mathbf{F}(\mathbf{y}), \mathbf{F}(\mathbf{z})\big) &= \sup\{d(f(y_{i-1}), f(z_{i-1})) : i \,\epsilon Z\} \\ &\leq \lambda \sup\{d(y_{i-1}, z_{i-1}) : i \,\epsilon Z\} \\ &= \lambda \sup\{d(y_i, z_i) : i \,\epsilon Z\} \leq \lambda D(\mathbf{y}, \mathbf{z}). \end{aligned}$$

By the Banach Contraction Principle, **F** has a fixed point **w** = $\{w_i\}_{i\epsilon Z} \epsilon$ **E** such that **F**(**w**) = **w** that is $w_i = f(w_{i-1})$ and $d(x_i, w_i) \leq \epsilon$ since **w**ϵ**E**. Thus, the trajectory **w** is ε-shadows the δ-pseudo trajectory $\{x_i\}_{i\epsilon Z}$. This completes the proof. □

Proof of Shadowing Lemma: If a linear homeomorphism f is hyperbolic, then $f = f_1 \oplus f_2$, where both f_1 and ${f_2}^{-1}$ are contractions. By Proposition 2.3.1 both f_1 and ${f_2}^{-1}$ have the shadowing property. Moreover, Lemma 2.3.2 implies that f_2 has the shadowing property and finally by Lemma 2.3.1 f has the shadowing property. This completes the proof of the shadowing lemma. □

Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f: X \to X$ and $g: Y \to Y$ be two maps on X and Y respectively. We say that f and g are topologically conjugate if there exist a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$. In particular, if h and h^{-1} are uniformly continuous then f and g are said to be uniformly conjugate.

Next, we modify the proof of Theorem 22 in (Thkkar and Das, 2014 a) for nonautonomous system to prove the following theorem.

Theorem 2.3.1: Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to X$ and $g: Y \to Y$ be two continuous maps. If f and g are uniformly conjugate then f has the shadowing property if and only if g has the shadowing property.

Proof: Given $\varepsilon > 0$, and let $h: X \to Y$ be a homeomorphism so that $g \circ h = h \circ f$. Applying the uniform continuity of h implies that there exists $0 < \varepsilon_1 < \varepsilon$ such that, for any $x_1, x_2 \epsilon X$ with $d_X(x_1, x_2) < \varepsilon_1$, $d_Y(h(x_1), h(x_2)) < \varepsilon$. As f has the shadowing property there exists $\delta_1 > 0$ such that every δ_1-pseudo trajectory of f is ε_1-traced by some point in X. Noting the fact that h^{-1} is uniformly continuous, there exist $0 < \delta < \delta_1$ such that, for any $y_1, y_2 \epsilon Y$ with $d_Y(y_1, y_2) < \delta$, $d_X(h^{-1}(y_1), h^{-1}(y_2) < \delta_1$. Now we claim that every δ-pseudo trajectory of g is ε-traced by some point of Y. In fact, for any δ-pseudo trajectory $\{y_i\}_{i=1}^{\infty}$ of g, applying

$$d_X(f(h^{-1}(y_i)), h^{-1}(y_{i+1})) = d_X(h^{-1}(g(y_i)), h^{-1}(y_{i+1})) < \delta_1.$$

It follows that $\{x_i\}_{i=1}^{\infty} = \{h^{-1}(y_i)\}_{i=1}^{\infty}$ is a δ_1-pseudo trajectory of f. Then there exist $x \epsilon X$ such that $\{x_i\}_{i=1}^{\infty}$ is ε_1-traced by x. This implies that

$$d_Y(g^i(h(x)), y_i) = d_Y(h(f^i(x)), h(h^{-1}(y_i))) < \varepsilon.$$

Thus, the proof of this theorem is complete. □

2.4 Bi-shadowing property

The shadowing or pseudo orbit tracing property of dynamical systems as we mentioned in the last section is often used to justify the validity of computer simulation of the system, asserting that there is a true trajectory of the system close to the computed pseudo trajectory. From this it is often concluded that the behavior of computed system reflects that of the original system.

The inverse question as to whether every true trajectory can be shadowed by some pseudo trajectory is of no less practical importance for complete understanding the relationship between true trajectories and pseudo trajectories. This is the idea of inverse shadowing. While any pseudo trajectory for some δ is in principle possible, in practice only those that belong to some particular class τ may in fact occur. Typically, only general characteristics of such pseudo trajectories will be known rather than a complete definition of τ itself. The problem of inverse shadowing with respect to such class τ is, nevertheless, still meaningful: Can every true trajectory of the given system $f: X \to X$ be shadowed by some trajectory from τ ? For instance, in the classical shadowing (direct shadowing) this class consists of all pseudo trajectories of the given system, while in inverse shadowing a natural and convenient class consists of trajectories of all continuous mappings φ that are sufficiently close to f.

Let $X = R^n$ and let $\mathbf{Tr}(f, K, \delta)$ denote the totality of finite or infinite δ-pseudo trajectories of f belong entirely to $K \subseteq X$. Since a true trajectory can be regarded as a $\delta = 0$ pseudo trajectory, the set of all finite or infinite trajectories which belong entirely to K will be denoted by $\mathbf{Tr}(f, K, 0)$. Obviously a true trajectory is also a δ-pseudo trajectory for any $\delta > 0$, so $\mathbf{Tr}(f, K, 0) \subset \mathbf{Tr}(f, K, \delta)$ where the inclusion is strict because obviously not every pseudo trajectory is a true trajectory. For the distance between the maps φ and f on X we will use

$$\|\varphi - f\|_\infty = \sup_{x \epsilon X} \|\varphi(x) - f(x)\|.$$

Definition 2.4.1 (Al-Nayef, 1997): A dynamical system generated by a mapping $f: X \to X$ is said to be bi-shadowing on a subset K of X with positive parameters α and β if for any given pseudo-trajectory $\boldsymbol{y} = \{y_n\} \epsilon$ $\mathbf{Tr}(f, K, \delta)$ with $0 \le \delta \le \beta$ and any continuous mapping $\varphi: X \to X$ satisfing

$$\delta + \|\varphi - f\|_\infty \le \beta \tag{2.4}$$

there exists a trajectory $\mathbf{x} = \{x_n\} \epsilon$ **Tr** $(\varphi, X, 0)$ such that

$$\|x_n - y_n\| \le \alpha(\delta + \|\varphi - f\|_\infty) \tag{2.5}$$

for all n for which $\boldsymbol{y}$ is defined.

Bi-shadowing includes the definition of both the direct shadowing and the inverse shadowing: taking $\varphi = f$ in (2.4) and (2.5) gives $\alpha\delta$-shadowing of any δ-pseudo trajectory $\boldsymbol{y}\,\epsilon$ **Tr**(f, K, δ) by a true trajectory $\mathbf{x}\,\epsilon$ **Tr**$(f, K, 0)$, while inverse shadowing follows because a trajectory $\mathbf{x}\,\epsilon$ **Tr**$(\varphi, X, 0)$ can always be found which shadows a given true trajectory $\boldsymbol{y}\,\epsilon$ **Tr**$(f, K, 0)$, considered here as the δ-pseudo trajectory with $\delta = 0$.

Cyclic or periodic behavior is of particular importance in dynamical systems. A trajectory $\mathbf{x} = \{x_n\}_{n=0}^{N} \epsilon$ **Tr**$(f, K, 0)$ is called a cycle of period N if $x_N = x_0$. Analogously, a pseudo trajectory of $\boldsymbol{y} = \{y_n\}_{n=0}^{N} \epsilon$ **Tr**(f, K, δ) will be called a δ-pseudo trajectory of period N if $|y_n - y_0| \le \delta$. Let $C(f, K, \delta) \subset$ **Tr**(f, K, δ) denoting the totality of δ-pseudo cycles of any period belonging entirely to the subset K of X, with $C(f, K, 0) \subset$ **Tr**$(f, K, 0)$ denoting the totality of proper cycles of any period which are contained entirely in K. Obviously $C(f, K, 0) \subset C(f, K, \delta)$ for every $\delta > 0$.

Definition 2.4.2 (Al-Nayef, 1997): A dynamical system generated by a mapping $f: X \to X$ where X is an open subset of R^n is said to be cyclically bi-shadowing on a subset K of X with positive parameters α and β if for any given pseudo-cycle $\boldsymbol{y}\,\epsilon\, C(f, K, \delta)$ with $0 \le \delta \le \beta$ and any continuous mapping $\varphi: X \to X$ satisfying (2.4) there exists a proper cycle $\mathbf{x}\,\epsilon\, C(\varphi, X, 0)$ of period N equal to that of $\boldsymbol{y}$ such that (2.5) holds for $n = 0,1, \ldots, N$.

Note that the cycle $\mathbf{x}$ here is required only to be in X rather than in the subset K.

CHAPTER THREE
Bi-Shadowing Properties of Discrete Systems

In this chapter, we discuss the bi-shadowing properties satisfied by discrete systems under certain conditions, we also present and prove the main results of the thesis.

A metric space (X, d) is called compact if for every covering $(\mathcal{U}_\alpha)_{\alpha\epsilon\Delta}$ of X by open sets (open covering) there exist a finite subfamily $(\mathcal{U}_\alpha)_{\alpha\epsilon H}$ ($H \subset \Delta$ and finite) which is a covering of X.

Throughout this chapter, we assume (X, d) to be a compact metric space.

3.1 Bi-Shadowing of Almost Contraction.

In this section we state and prove some theorems regarding bi-shadowing contractions and almost contractions.

A mapping $f: X \to X$ is called (λ, L)-contraction or almost contraction, if there exist constants $0 \leq \lambda < 1$ and $L \geq 0$ such that

$$d\big(f(x), f(y)\big) \leq \lambda\, d(x, y) + L\, d\big(y, f(x)\big) \qquad \text{for all } x, y \epsilon X.$$

The almost contraction condition above implicitly includes the following dual one

$$d\big(f(x), f(y)\big) \leq \lambda\, d(x, y) + L\, d\big(x, f(y)\big) \qquad \text{for all } x, y \epsilon X.$$

Obviously, any q-contractive is almost contraction with $\lambda = q$ and $L = 0$.

We now give the definition of the concept of bi-shadowing in the context of a general metric space.

Definition 3.1.1 (Al-Badarneh, 2015 b): A continuous mapping $f: X \to X$, where X is a metric space, is called bi-shadowing with respect to a comparison class $C(X)$ consisting of continuous mappings on X and with positive parameters α and β if for any given δ-pseudo trajectory $\{y_i\}_{i=0}^{\infty}$ of f with $0 \leq \delta \leq \beta$ and any $\varphi\, \epsilon\, C(X)$ satisfying

$$\delta + \sup_{x \epsilon X} d(\varphi(x), f(x)) \leq \beta$$

there exists a true trajectory $\{w_i\}_{i=0}^{\infty}$ of φ such that

$$d(w_i, y_i) \leq \alpha(\delta + \sup_{x \epsilon X} d(\varphi(x), f(x))), \quad i = 0, 1, 2, \dots .$$

Theorem 3.1.1 (Al-Badarneh, 2015 b): Let (X, d) be a metric space and $f: X \to X$ a λ-contractive mapping on X, that is, there exists a constant $0 < \lambda < 1$ such that

$$d((f(x), f(y)) \leq \lambda\, d(x, y), \quad \text{for all } x, y \in X.$$

Then f is bi-shadowing on X with respect to the comparison class $C(X)$ and with positive parameters α and β given by

$$\alpha = \frac{2}{1-\lambda} \quad \text{and} \quad \beta = (1-\lambda). \tag{3.1}$$

Proof: Fix $\delta < (1-\lambda)/2$. Let $\{y_i\}_{i=0}^{\infty}$ be a given δ-pseudo trajectory of f satisfying

$$d(f(y_i), y_{i+1}) < \delta \qquad i = 0, 1, 2, \ldots$$

and let $g: X \to X$ be any continuous mapping such that

$$\sup_{x \in X} d\big(g(x), f(x)\big) \leq (1-\lambda)/2.$$

It follows that

$$\delta + \sup_{x \in X} d\big(g(x), f(x)\big) \leq (1-\lambda)/2 \; + (1-\lambda)/2 = \beta.$$

Consider a true trajectory $\{x_i\}_{i=0}^{\infty}$ for the mapping g such that $g(x_i) = x_{i+1}$ for $i = 0, 1, 2, \ldots$ and satisfying the relation

$$d(x_0, y_0) \leq \frac{1}{1-\lambda}\Big(\delta + \sup_{x \in X} d(g(x), f(x))\Big) \tag{3.2}$$

For $i = 1$ we have:

$$\begin{aligned} d(x_1, y_1) &\leq d(g(x_0), f(y_0)) + d(f(y_0), y_1) \\ &\leq d\big(g(x_0), f(x_0)\big) + d\big(f(x_0), f(y_0)\big) + d(f(y_0), y_1) \\ &\leq \Big(\delta + \sup_{x \in X} d(g(x), f(x))\Big) + \lambda d(x_0, y_0). \end{aligned}$$

For $i = 2$ we have:

$$\begin{aligned} d(x_2, y_2) &\leq d(g(x_1), f(x_1)) + d(f(x_1), f(y_1)) + d(f(y_1), y_2) \\ &\leq (1+\lambda)\Big(\delta + \sup_{x \in X} d(g(x), f(x))\Big) + \lambda^2 d(x_0, y_0). \end{aligned}$$

For i =3 we have:

$$d(x_3, y_3) \leq d(g(x_2, f(x_2)) + d(f(x_2), f(y_2)) + d(f(y_2), y_3)$$

$$\leq (1 + \lambda + \lambda^2)(\delta + \sup_{x \epsilon X} d(g(x), f(x))) + \lambda^3 d(x_0, y_0).$$

By induction, we obtain

$$d(x_i, y_i) \leq (1 + \lambda + \lambda^2 + \cdots + \lambda^{i-1})(\delta + \sup_{x \epsilon X} d(g(x), f(x))) + \lambda^i d(x_0, y_0)$$

$$\leq \left(\frac{1}{1-\lambda}\right)(\delta + \sup_{x \epsilon X} d(g(x), f(x))) + \ d(x_0, y_0).$$

Since λ< 1 and using the condition (3.1) and(3.2) we have

$$d(x_i, y_i) \leq \frac{2}{1-\lambda} (\delta + \sup_{x \epsilon X} d(g(x), f(x)))$$

$$= \alpha(\delta + \sup_{x \epsilon X} d(g(x), f(x))).$$

Thus, the proof of this theorem is complete. □

Theorem 3.1.2 (Al-Badarneh, 2015 b): Let (X, d) be a metric space and $f: X \to X$ a continuous almost contractive mapping, that is there exist constants $0 < \lambda < 1$ and $L \geq 0$ such that

$$d\big(f(x), f(y)\big) \leq \lambda d(x, y) + L d\big(y, f(x)\big) \quad \text{for all } x, y \, \epsilon \, X.$$

Assume that $\lambda + L < 1$, and that f is satisfying the following conditions:

i) For every δ-pseudo trajectory $\{z_i\}_{i=0}^{\infty} \subseteq X$ of f with $\delta < (1 - \lambda - L)/2$ the following series is convergent: $S := \sum_{i=0}^{\infty} d(f(z_i), z_i)$.

ii) For every continuous mapping $g: X \to X$ satisfying

$$\sup_{x \epsilon X} d(g(x), f(x)) \leq (1 - \lambda - L)/2,$$

the following inequality is satisfied:

$$LS < \delta + \sup_{x \epsilon X} d(g(x), f(x)). \tag{3.3}$$

Then f is bi-shadowing on X with respect to the class $C(X)$ and with parameters α and β given by

$$\alpha = \frac{2}{1 - \lambda - L} \quad \text{and} \quad \beta = (1 - \lambda - L) \tag{3.4}$$

Proof: Fix $\delta < (1-\lambda-L)/2$ and $\{y_i\}_{i=0}^{\infty}$ be a δ-pseudo trajectory of f satisfying

$$d(f(y_i), y_{i+1}) < \delta \qquad i = 0,1,2,\dots .$$

Let also $g: X \to X$ be any continuous mapping such that

$$d_\infty := \sup_{x \epsilon X} d(g(x), f(x)) \leq (1-\lambda-L)/2$$

It follows that

$$\delta + \sup_{x \epsilon X} d(g(x), f(x)) \leq \beta.$$

We consider the true trajectory $\{w_i\}_{i=0}^{\infty}$ for the mapping g such that $w_{i+1} = g(w_i)$ for $i = 0,1,2,\dots$. We use the relation (3.3) to choose w_0 with the following property:

$$d(w_0, y_0) + \frac{LS}{1-\lambda-L} \leq \frac{\delta + d_\infty}{1-\lambda-L} \tag{3.5}$$

For the case $i = 1$ we have:

$$\begin{aligned} d(w_1, y_1) &\leq d\big(g(w_0), f(y_0)\big) + d(f(y_0), y_1) \\ &\leq d\big(g(w_0), f(w_0)\big) + d\big(f(w_0), f(y_0)\big) + d(f(y_0), y_1) \\ &\leq (\delta + d_\infty) + (\lambda + L) d(w_0, y_0) + L d\big(y_0, f(y_0)\big). \end{aligned}$$

For the case $i = 2$ we have:

$$\begin{aligned} d(w_2, y_2) &\leq d\big(g(w_1), f(y_1)\big) + d\big(f(y_1), y_2)\big) \\ &\leq (\delta + d_\infty) + \lambda d(w_1, y_1) + L\big[d(w_1, y_1) + d\big(y_1, f(y_1)\big)\big] \\ &\leq (\delta + d_\infty) + (\lambda + L) d(w_1, y_1) + L d\big(y_1, f(y_1)\big) \\ &\leq \big(1 + (\lambda + L)\big)(\delta + d_\infty) + (\lambda + L)^2 d(w_0, y_0) \\ &\qquad + L(\lambda + L) d\big(y_0, f(y_0)\big) + L d\big(y_1, f(y_1)\big). \end{aligned}$$

Similarly, for $i = 3$ we have:

$$\begin{aligned} d(w_3, y_3) &\leq d(w_3, f(w_2)) + d(f(w_2), y_3) \\ &\leq (\delta + d_\infty) + (\lambda + L) d(w_2, y_2) + L d\big(y_2, f(y_2)\big) \\ &\leq (1 + (\lambda + L) + (\lambda + L)^2)(\delta + d_\infty) + (\lambda + L)^3 d(w_0, y_0) \\ &\quad + L(\lambda + L)^2 d\big(y_0, f(y_0)\big) + L(\lambda + L) d\big(y_1, f(y_1)\big) + L d\big(y_2, f(y_2)\big). \end{aligned}$$

In general, we obtain

$$d(w_i, y_i) \leq (\delta + d_\infty) \sum_{k=0}^{i-1} (\lambda + L)^k + (\lambda + L)^i d(w_0\,, y_0) \\ + \sum_{k=0}^{i-1} L(\lambda + L)^{i-k-1} d(y_k, f(y_k)).$$

Note that, if we write $\sum_{k=0}^{i-1} L(\lambda + L)^{i-k-1} d\big(y_k, f(y_k)\big) = \sum_{k=0}^{i-1} a_k\, b_{i-k-1}$, where $a_i = d\big(y_i, f(y_i)\big)$, and $b_i = L(\lambda + L)^i$ for $i = 0,1,2, \ldots$. If $c_i = \sum_{k=0}^{i-1} a_k b_{i-k-1}$, $i = 1, 2, \ldots$ then Theorem 8.46 in (Apostol, 1978) implies that the series $\sum_{i=1}^{\infty} c_i$ is convergent and $\sum_{i=1}^{\infty} c_i = \frac{LS}{1-\lambda-L}$. Therefore, by the condition (3.4) and (3.5) and since $\lambda + L < 1$ we have

$$\begin{aligned} d(w_i, y_i) &\leq (\delta + d_\infty) \sum_{i=0}^{\infty} (\lambda + L)^i + (\lambda + L)^i\, d(w_0, y_0) + \sum_{i=1}^{\infty} c_i \\ &\leq \frac{\delta + d_\infty}{1 - \lambda - L} + d(w_0, y_0) + \frac{LS}{1 - \lambda - L} \\ &\leq \frac{\delta + d_\infty}{1 - \lambda - L} + \frac{\delta + d_\infty}{1 - \lambda - L} = \alpha(\delta + d_\infty). \end{aligned}$$

Thus, the proof of this theorem is complete. □

Definition 3.1.2 (Berinda, 2004): A mapping $f\colon X \to X$ is called Kannan mapping if there exists a constant $0 < a < 1/2$ such that

$$d\big(f(x), f(y)\big) \leq a\big[d\big(x, f(x)\big) + d\big(y, f(y)\big)\big], \text{ for all } x, y \in X. \tag{3.6}$$

Definition 3.1.3 (Berinda, 2004): A mapping $f\colon X \to X$ is called Chatterjea mapping if there exists a constant $0 < c < 1/2$ such that

$$d\big(f(x), f(y)\big) \leq c\big[d\big(x, f(y)\big) + d\big(y, f(x)\big)\big], \text{ for all } x, y \in X. \tag{3.7}$$

It was shown in (Berinda, 2004) that Kannan mappings and Chatterjea mappings are almost contractions.

Theorem 3.1.3 (Berinda, 2004): Let (X, d) be a metric space. Then

a) A Kannan mapping $f\colon X \to X$ with $0 < a < 1/2$ is almost contraction with constants

$$\lambda = \frac{a}{1 - a} \quad \text{and} \quad L = \frac{2a}{1 - a}\,. \tag{3.8}$$

b) A Chatterjea mapping $f\colon X \to X$ with $0 < c < 1/2$ is almost contraction with constants

$$\lambda = \frac{c}{1 - c} \quad \text{and} \quad L = \frac{2c}{1 - c}\,. \tag{3.9}$$

Theorem 3.1.4 (Al-Badarneh, 2015 b): Let a continuous mapping $f: X \to X$ be a Kannan map- ing such that the conditions i) and ii) of Theorem 3.1.2 are satisfied. Then f is bi-shadowing on X with respect to the class $C(X)$ provided that $a < 1/4$ and with parameters α and β given by:

$$\alpha = \frac{2 - 2a}{1 - 4a} \quad \text{and} \quad \beta = \frac{1 - 4a}{1 - a} \tag{3.10}$$

Proof: If f is a continuous Kannan mapping satisfying the condition i) and ii) of Theorem 3.1.2 and since by Theorem 3.1.3 a Kannan mapping is almost contraction then Theorem 3.1.2 implies that f is bi-shadowing on X provided that

$$\lambda + L = \frac{3a}{1 - a} < 1$$

that is $a < 1/4$. Moreover, the values of α and β in (3.10) are obtained by substituting the values of λ and L of (3.8) in (3.4). □

Similar argument can be used to prove the next theorem. Since a Chtterjea mapping is almost contraction.

Theorem 3.1.5 (Al-Badarneh, 2015 b): Let a continuous mapping $f: X \to X$ be Chtterjea map- ing satisfying the conditions i) and ii) of Theorem (3.1.2). Then f is bi- shadowing on X with respect to the class $C(X)$ provided that $c < 1/4$ and with parameters α and β given by:

$$\alpha = \frac{2 - 2c}{1 - 4c} \quad \text{and} \quad \beta = \frac{1 - 4c}{1 - c}.$$

Definition 3.1.4 (Reich, 1971): A mapping $f: X \to X$ is called Reich mapping if there exist constants $a, b, c \geq 0$ with $a + b + c < 1$ such that

$$d\big(f(x), f(y)\big) \leq ad(x, y) + bd\big(x, f(x)\big) + cd\big(y, f(y)\big),$$

for all $x, y \in X$.

Theorem 3.1.6 (Pacurar M. and Pacurar R.V., 2007): Let (X, d) be a metric space and $f: X \to X$ a Reich mapping with constants $a, b, c \geq 0$ such that $a + b + c < 1$. Then f is almost contraction with constants

$$\lambda = \frac{a + b}{1 - c} \quad \text{and} \quad L = \frac{b + c}{1 - c}. \tag{3.11}$$

Theorem 3.1.7 (Al-Badarneh, 2015 b): Let (X, d) be a metric space and let $f: X \to X$ be a continuous Reich mapping with constants $a, b, c \geq 0$ such that $a + b + c < 1$ and assume that the condition i) and ii) of Theorem 3.1.2 are satisfied. Then f is bi-shadowing on X with respect to the class $C(X)$ provided that $a + 2b + 2c < 1$ and with parameters α and β given by:

$$\alpha = \frac{2 - 2c}{1 - a - 2b - 2c} \quad \text{and} \quad \beta = \frac{1 - a - 2b - 2c}{1 - c}. \tag{3.12}$$

Proof: Let $f: X \to X$ be a continuous Reich mapping with constants $a, b, c \geq 0$ such that $a + b + c < 1$. It follows by Theorem 3.1.6 that f is almost contraction with constant λ and L given in (3.11). Thus Theorem 3.1.2 implies that f is bi-shadowing on X with respect to the class $C(X)$ provided that

$$\lambda + L = \frac{a + 2b + c}{1 - c} < 1$$

that is $a + 2b + 2c < 1$. The value of α and β in (3.12) can be obtained easily by substituting the values of λ and L of (3.11) in (3.4). □

One of the most general contraction condition has been obtained by (Ciric, 1974): there exists $0 < h < 1$ such that

$$d\big(f(x), f(y)\big) \leq h\, M(x, y) \quad \text{for all } x, y \in X, \tag{3.13}$$

where

$$M(x, y) = \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}.$$

A mapping $f: X \to X$ satisfying (3.13) is commonly called quasi contraction.

Theorem 3.1.8 (Berinda, 2004): Any quasi-contraction with $0 < h < 1/2$ is an almost contraction.

By combining Theorems 3.1.8 and 3.1.2 we have the following result.

Theorem 3.1.9 (Al-Badarneh, 2015 a): Let a continuous mapping $f: X \to X$ be quasi-contraction with $0 < h < 1/2$ and assume that the conditions i) and ii) of Theorem 3.1.2 are satisfied. Then f is bi-shadowing on X with respect to the class $C(X)$.

Proof: For a continuous mapping $f: X \to X$, Theorem 3.1.8 implies that f is almost contraction with appropriate value λ and L. It follows from the proof of Proposition 3 of (Berinda, 2004) that the values of λ and L depend on what the maximum in (3.13) is. For example, if $M(x, y) = d\big(x, f(y)\big)$, then

$$d\big(f(x),f(y)\big) \le \frac{h}{1-h}d(x,y) + \frac{h}{1-h}d\big(y,f(x)\big).$$

Thus $\lambda = L = \frac{h}{1-h}$ and since the condition i) and ii) of Theorem (3.1.2) are assumed to be satisfied, Theorem 3.1.2 implies that f is bi-shadowing with respect to the class $C(X)$ provided that $\lambda + L = \frac{2h}{1-h} < 1$, that is $h < 1/3$. In this case, the values of α and β are

$$\alpha = \frac{2(1-h)}{1-3h} \quad \text{and} \quad \beta = \frac{1-3h}{1-h}.$$

The other cases are treated similarly. The proof is complete. □

Definition3.1.5 (Ciric, 1971): A mapping $f\colon X \to X$ is called generalised contraction, or Ciric contraction, if there exist nonnegative constants α, β,γ and δ with $\alpha + \beta + \gamma + 2\delta < 1$ such that

$$d\big(f(x),f(y)\big) \le \alpha d(x,y) + \beta d\big(x,f(x)\big) + \gamma d\big(y,f(y)\big) \\ +\delta\big[d\big(x,f(y)\big) + d\big(y,f(x)\big)\big]$$

for all $x,y \in X$.

Lemma 3.1.1 (Al-Badarneh, 2015 a): Let $f\colon X \to X$ be a generalised contraction (Ciric) with $\alpha + \beta + \gamma + 2\delta < 1$. Then f is almost contraction with constants

$$\lambda = \frac{\alpha+\beta+\delta}{1-\gamma-\delta} \quad \text{and} \quad L = \frac{\beta+\gamma+2\delta}{1-\gamma-\delta}$$

Proof: $d\big(f(x),f(y)\big) \le \alpha d(x,y) + \beta d\big(x,f(x)\big) + \gamma d\big(y,f(y)\big)$
$$+\delta d\big(x,f(y)\big) + \delta d\big(y,f(x)\big)$$

$$\le \alpha d(x,y) + \beta\big[d(x,y) + d\big(y,f(x)\big)\big] + \delta\big[d(x,y) + d\big(y,f(y)\big)\big] \\ +\delta d\big(y,f(x)\big) + \gamma d\big(y,f(y)\big)$$

$$= (\alpha+\beta+\delta)d(x,y) + (\delta+\gamma)d\big(y,f(y)\big) + (\beta+\delta)d\big(y,f(x)\big)$$

$$\le (\alpha+\beta+\delta)d(x,y) + (\delta+\gamma)d\big(y,f(x)\big) + (\delta+\gamma)d\big(f(x),f(y)\big) \\ +(\beta+\delta)d\big(y,f(x)\big)$$

$$= (\alpha+\beta+\delta)d(x,y) + (\beta+\gamma+2\delta)d\big(y,f(x)\big) + (\gamma+\delta)d\big(f(x),f(y)\big).$$

It follows that

$$d\big(f(x),f(y)\big) \le \frac{\alpha+\beta+\delta}{1-\gamma-\delta}d(x,y) + \frac{\beta+\gamma+2\delta}{1-\gamma-\delta}d\big(y,f(x)\big).$$

If we take

$$\lambda = \frac{\alpha + \beta + \delta}{1 - \gamma - \delta} \quad \text{and} \quad L = \frac{\beta + \gamma + 2\delta}{1 - \gamma - \delta},$$

then clearly, $L \geq 0$ and $\lambda < 1$ since by assumption $\alpha + \beta + \gamma + 2\delta < 1$. Hence f is almost contraction. □

By using this Lemma and Theorem 3.1.2 we have the following result.

Theorem 3.1.10: (Al-Badarneh, 2015 a): Let a continuous mapping $f: X \to X$ be a generalised (Ciric) contraction and assume that the conditions i) and ii) of Theorem 3.1.2 are satisfied. Then f is bi-shadowing on X with respect to the class $C(X)$ provided that $\alpha + 2\beta + 2\gamma + 4\delta < 1$ and with positive parameters:

$$\alpha = \frac{2(1 - \gamma - \delta)}{1 - 2\gamma - 4\delta - \alpha - 2\beta} \quad \text{and} \quad \beta = \frac{1 - 2\gamma - 4\delta - \alpha - 2\beta}{1 - \gamma - \delta}.$$

Proof: If f is a generalised contraction such that the condition i) and ii) of Theorem 3.1.2 are satisfied and since by Lemma 3.1.1 a generalised contraction is almost contraction, Theorem 3.1.2 implies that f is bi-shadowing on X with respect to the class $C(X)$ for the values of λ and L satisfying the relation

$$\lambda + L = \frac{\alpha + 2\beta + \gamma + 3\delta}{1 - \gamma - \delta} < 1,$$

since $\alpha + 2\beta + 2\gamma + 4\delta < 1$. □

3.2 More Results on Bi-Shadowing

In this section, we introduce more results on bi-shadowing and suggest a definition of new property of bi-shadowing.

Let (X, d_X) and (Y, d_Y) be two metric spaces and define a metric d on $X \times Y$ by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Let $f: X \to X$ and $g: Y \to Y$ be continuous mappings, and let $f \times g$ be a map on $X \times Y$ defined by $(f \times g)(x, y) = \big(f(x), g(y)\big)$.

Theorem 3.2.1: If f and g have the bi-shadowing property with respect to the comparison classes $C(X)$ and $C(Y)$ with positive parameters α and β, then $f \times g$ has the bi-shadowing property with respect to the class $C(X) \times C(Y)$ with the same parameters α and β.

Proof: Let $\{(x_i, y_i)\}_{i=1}^{\infty}$ be a δ-pseudo trajectory of the map $f \times g$, with $0 \le \delta \le \beta$, and let $\phi(x) \times \varphi(y) \epsilon\, C(X) \times C(Y)$ satisfying

$$\delta + \sup_{(x,y)\epsilon X\times Y} d\big((f \times g)(x, y), (\phi \times \varphi)(x, y)\big) \le \beta.$$

Since $\{(x_i, y_i)\}_{i=1}^{\infty}$ is a δ-pseudo trajectory of the map $f \times g$, then we have

$$\begin{aligned} d\big((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})\big) &= d\Big(\big(f(x_i), g(y_i)\big), (x_{i+1}, y_{i+1})\Big) \\ &= \max\{d_X(f(x_i), x_{i+1}), d_Y(g(y_i), y_{i+1})\} \\ &< \delta. \end{aligned}$$

Thus, $d_X(f(x_i), x_{i+1}) < \delta$ and $d_Y(g(y_i), y_{i+1}) < \delta$, which implies that both $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are δ-pseudo trajectories of f and g respectively. But f and g have the bi-shadowing property, hence for any $\phi \,\epsilon\, C(X)$ and $\varphi \epsilon\, C(Y)$ satisfying

$$\delta + \sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) \le \beta$$

and

$$\delta + \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big) \le \beta,$$

there exist true trajectories $\{w_i\}_{i=1}^{\infty}$ of ϕ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i, w_i) \le \alpha(\delta + \sup_{x\epsilon X} d_X(f(x), \phi(x))) \tag{3.14}$$

and

$$d_Y(y_i, z_i) \le \alpha(\delta + \sup_{y\epsilon Y} d_Y(g(y), \varphi(y))). \tag{3.15}$$

for $i \,\epsilon\, Z^+$.

Without lost of generality, we may assume that

$$\sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) > \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big),$$

then for $i \,\epsilon\, Z^+$, we have the following three cases:

1) For the values of $i \,\epsilon\, Z^+$ for which $d_X(x_i, w_i) > d_Y(y_i, z_i)$, and using (3.14), we have

$$\begin{aligned} d\big((x_i, y_i), (w_i, z_i)\big) &= \max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} \\ &\le \alpha(\delta + \sup_{x\epsilon X} d_X(f(x), \phi(x))), \end{aligned}$$

so, for every $x\epsilon X$, $y\epsilon Y$ we have

$$d\big((x_i, y_i), (w_i, z_i)\big) \leq \alpha(\delta + \sup_{x \epsilon X} d_X(f(x), \phi(x)))$$
$$= \alpha(\delta + \max\{\sup_{x \epsilon X} d_X(f(x), \phi(x)), \sup_{y \epsilon Y} d_Y(g(y), \varphi(y))\})$$
$$= \alpha(\delta + \sup_{x \epsilon X, y \epsilon Y} (\max\{d_X(f(x), \phi(x)), d_Y(g(y), \varphi(y))\}))$$
$$= \alpha(\delta + \sup_{(x,y) \epsilon X \times Y} d((f \times g)(x, y), (\phi \times \varphi)(x, y))).$$

Note that $\{(w_i, z_i)\}_{i=1}^{\infty}$is a true trajectory of $\phi \times \varphi$ since

$$(\phi \times \varphi)(w_i, z_i) = \big(\phi(w_i), \varphi(z_i)\big) = (w_{i+1}, z_{i+1}).$$

2) For the values of $i \epsilon Z^+$ for which $d_X(x_i, w_i) < d_Y(y_i, z_i)$, and by using (3.15), we have

$$d\big((x_i, y_i), (w_i, z_i)\big) = \max\{d_X(x_i, w_i), d_Y(y_i, z_i)\}$$
$$\leq \alpha(\delta + \sup_{y \epsilon Y} d_Y(g(y), \varphi(y)))$$
$$\leq \alpha\ (\delta + \sup_{x \epsilon X} d_X(f(x), \phi(x))).$$

From the argument of case (1) above we obtain

$$d\big((x_i, y_i), (w_i, z_i)\big) \leq \alpha(\delta + \sup_{(x,y) \epsilon X \times Y} d((f \times g)(x, y), (\phi \times \varphi)(x, y))).$$

3) For the values of $i \epsilon Z^+$ for which $d_X(x_i, w_i) = d_Y(y_i, z_i)$, we have the same result in (1) and (2).

By combining the three cases, we have

$$d((x_i, y_i), (w_i, z_i)) \leq \alpha(\delta + \sup_{(x,y) \epsilon X \times Y} d((f \times g)(x, y), (\phi \times \varphi)(x, y))),$$

for all $i \in Z^+$.

which mean that $f \times g$ has the bi-shadowing property. □

Theorem 3.2.2: Let $f: X \to X$ and $g: Y \to Y$ be continuous mappings. If $f \times g$ has the bi-shadowing property with respect to the comparison class $C(X) \times C(Y)$ and positive parameters α and β, then at least one of the mappings f and g has the bi-shadowing property on the class $C(X)$ and $C(Y)$ respectively with the same parameters α and β.

Proof: Let $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be δ-pseudo trajectories of f and g respectively, with $0 \leq \delta \leq \beta$, and let $\phi \epsilon C(X)$ and $\varphi \epsilon C(Y)$ satisfies

$$\delta + \sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) \leq \beta$$

and

$$\delta + \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big) \leq \beta.$$

Then

$$d\big((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})\big) = d((f(x_i), g(y_i)), (x_{i+1}, y_{i+1}))$$
$$= \max\{d_X(f(x_i), x_{i+1}), d_Y(g(y_i), y_{i+1})\} < \delta.$$

So, $d\big((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})\big) < \delta$ and the sequence $\{(x_i, y_i)\}_{i=1}^{\infty}$ is a δ-pseudo trajectory of $f \times g$.

But $f \times g$ has the bi-shadowing property, and hence any map $\phi \times \varphi \,\epsilon\, C(X) \times C(Y)$ satisfying

$$\delta + \sup_{(x,y)\epsilon X\times Y} d\big((f \times g)(x, y), (\phi \times \varphi)(x, y)\big) \leq \beta,$$

there exists a true trajectory $\{(w_i, z_i)\}_{i=1}^{\infty}$ of $\phi \times \varphi$ such that

$$d\big((x_i, y_i), (w_i, z_i)\big) \leq \alpha(\delta + \sup_{(x,y)\epsilon X\times Y} d((f \times g)(x, y), (\phi \times \varphi)(x, y))).$$

So, for every $x\epsilon X$, $y\epsilon Y$ we have

$$\max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} \leq \alpha(\delta + \sup_{(x,y)\epsilon X\times Y} d((f \times g)(x, y), (\phi \times \varphi)(x, y)))$$
$$= \alpha(\delta + \sup_{x\epsilon X, y\epsilon Y} (\max\{d_X(f(x), \phi(x)), d_Y(g(y), \varphi(y))\}))$$
$$= \alpha(\delta + \max\{\sup_{x\epsilon X} d_X(f(x), \phi(x)), \sup_{y\epsilon Y} d_Y(g(y), \varphi(y))\}).$$

Now, we have three cases:

1) If $\sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) > \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big)$, then we have

$d_X(x_i, w_i) \leq \alpha(\delta + \sup_{x\epsilon X} d_X(f(x), \phi(x)))$ and hence f has the bi-shadowing property with respect to $C(X)$.

2) If $\sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) < \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big)$, then we have

$d_Y(y_i, z_i) \leq \alpha(\delta + \sup_{y\epsilon Y} d_Y(g(x), \varphi(x)))$ and hence g has the bi-shadowing property with respect to $C(Y)$.

3) If $\sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) = \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big)$, then we have

$$d_X(x_i, w_i) \leq \alpha(\delta + \sup_{x\epsilon X} d_X(f(x), \phi(x)))$$

and

$$d_Y(y_i, z_i) \leq \alpha(\delta + \sup_{y\epsilon Y} d_X(g(y), \varphi(y)))$$

and hence both f and g have the bi-shadowing property.

Note that both $\{w_i\}_{i=1}^{\infty}$and $\{z_i\}_{i=1}^{\infty}$are true trajectories of ϕ and φ respectively since

$$\big(\phi(w_i), \varphi(z_i)\big) = (\phi \times \varphi)(w_i, z_i) = (w_{i+1}, z_{i+1}). \qquad \square$$

We now introduce the following definition of a bi-shadowing property for a pair of systems, which we call it mutually bi-shadowing.

Definition 3.2.1: Let $f: X \to X$ and $g: Y \to Y$ be continuous mappings. The pair of systems (f, g) is called mutually bi-shadowing with respect to the positive parameters α, β and comparison classes $C(X)$ and $C(Y)$ of f and g respectively, if for any given δ-pseudo trajectories $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ for f and g respectively, and for any $\phi \,\epsilon\, C(X)$ and $\varphi \epsilon\, C(Y)$ satisfies

$$\delta + \sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) \leq \beta$$

and

$$\delta + \sup_{y\epsilon Y} d_Y\big(g(y), \varphi(y)\big) \leq \beta,$$

there exist true trajectories $\{w_i\}_{i=1}^{\infty}$ of ϕ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i, w_i) \leq \alpha(\delta + \max\{\sup_{x\epsilon X} d_X(f(x), \phi(x)), \sup_{y\epsilon Y} d_Y(g(y), \varphi(y))\})$$

and

$$d_Y(y_i, z_i) \leq \alpha(\delta + \max\{\sup_{x\epsilon X} d_X(f(x), \phi(x)), \sup_{y\epsilon Y} d_Y(g(y), \varphi(y))\})$$

for every $x\epsilon X$, $y\epsilon Y$.

Now, in the context of this new definition of bi-shadowing for a pair of systems we have the following result, which is an improved version of Theorem 3.2.2.

Theorem 3.2.3: Let $f: X \to X$ and $g: Y \to Y$ be continuous mappings. Then the pair of systems (f, g) is mutually bi-shadowing with respect to the positive parameters α, β and comparison classes $C(X)$ and $C(Y)$ of f and g respectively, iff $f \times g$ has the bi-shadowing property with respect to the comparison class $C(X) \times C(Y)$ and the same parameters α and β.

Proof: Let $\{(x_i, y_i)\}_{i=1}^{\infty}$ be a δ-pseudo trajectory of the map $f \times g$, with $0 \le \delta \le \beta$, and let $\phi(x) \times \varphi(y) \in C(X) \times C(Y)$ be satisfying

$$\delta + \sup_{(x,y)\in X\times Y} d\big((f \times g)(x,y), (\phi \times \varphi)(x,y)\big) \le \beta.$$

Thus, both $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are δ-pseudo trajectories of f and g respectively. But the pair of systems (f, g) is mutually bi-shadowing, hence for any $\phi \in C(X)$ and $\varphi \in C(Y)$ be satisfying

$$\delta + \sup_{x\in X} d_X\big(f(x), \phi(x)\big) \le \beta$$

and

$$\delta + \sup_{y\in Y} d_Y\big(g(y), \varphi(y)\big) \le \beta,$$

there exists true trajectory $\{w_i\}_{i=1}^{\infty}$ of ϕ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i, w_i) \le \alpha(\delta + \max\{\sup_{x\in X} d_X(f(x), \phi(x)), \sup_{y\in Y} d_Y(g(y), \varphi(y))\})$$

and

$$d_Y(y_i, z_i) \le \alpha(\delta + \max\{\sup_{x\in X} d_X(f(x), \phi(x)), \sup_{y\in Y} d_Y(g(y), \varphi(y))\}).$$

Thus, from above we get for every $x\in X$, $y\in Y$

$$\begin{aligned}
\max\{d_X(x_i, w_i), d_Y(y_i, z_i)\}& \\
&\le \alpha(\delta + \max\{\sup_{x\in X} d_X(f(x), \phi(x)), \sup_{y\in Y} d_Y(g(y), \varphi(y))\}) \\
&= \alpha(\delta + \sup_{x\in X, y\in Y} (\max\{d_X(f(x), \phi(x)), d_Y(g(y), \varphi(y))\})) \\
&= \alpha(\delta + \sup_{(x,y)\in X\times Y} d((f \times g)(x,y), (\phi \times \varphi)(x,y))).
\end{aligned}$$

and hence

$$d\big((x_i, y_i), (w_i, z_i)\big) < \alpha(\delta + \sup_{(x,y)\in X\times Y} d((f \times g)(x,y), (\phi \times \varphi)(x,y))).$$

This shows that $f \times g$ has the bi-shadowing property with respect to the comparison class $C(X) \times C(Y)$ and the same parameters α and β.

Conversely, Let $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be δ-pseudo trajectories of f and g respectively, with $0 \le \delta \le \beta$, and let $\phi \in C(X)$ and $\varphi \in C(Y)$ be satisfying

$$\delta + \sup_{x\in X} d_X\big(f(x), \phi(x)\big) \le \beta$$

and

$$\delta + \sup_{y \epsilon Y} d_Y\big(g(y), \varphi(y)\big) \leq \beta.$$

Thus, $\{(x_i, y_i)\}_{i=1}^{\infty}$ is a δ-pseudo trajectory of $f \times g$. But $f \times g$ has the bi-shadowing property, and hence for any map $\phi \times \varphi \,\epsilon\, C(X) \times C(Y)$ satisfying

$$\delta + \sup_{(x,y)\epsilon X \times Y} d\big((f \times g)(x, y), (\phi \times \varphi)(x, y)\big) \leq \beta$$

there exists a true trajectory $\{(w_i, z_i)\}_{i=1}^{\infty}$ of $\phi \times \varphi$ such that

$$d\big((x_i, y_i), (w_i, z_i)\big) \leq \alpha(\delta + \sup_{(x,y)\epsilon X \times Y} d((f \times g)(x, y), (\phi \times \varphi)(x, y))).$$

So, we have

$$\max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} \leq$$
$$\alpha(\delta + \max\{\sup_{x \epsilon X} d_X(f(x), \phi(x)), \sup_{y \epsilon Y} d_Y(g(y), \varphi(y))\})$$

which implies that

$$d_X(x_i, w_i) \leq \alpha(\delta + \max\{\sup_{x \epsilon X} d_X(f(x), \phi(x)), \sup_{y \epsilon Y} d_Y(g(y), \varphi(y))\})$$

and

$$d_Y(y_i, z_i) \leq \alpha(\delta + \max\{\sup_{x \epsilon X} d_X(f(x), \phi(x)), \sup_{y \epsilon Y} d_Y(g(y), \varphi(y))\}).$$

Hence, the pair of systems (f, g) is mutually bi-shadowing. □

It is evident that Definition 3.2.1 implies the following two results:

(1) If both f and g have the bi-shadowing property, then the pair of systems (f, g) is mutually bi-shadowing.

(2) If the pair of systems (f, g) is mutually bi-shadowing, then at least f or g has the bi-shadowing property.

Theorem 3.2.4: Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f: X \to X$ and $g: Y \to Y$ be continuous mappings topologically conjugate by $h: X \to Y$. If there exists $\lambda \geq 1$ such that

$$d_X(x_1, x_2) \leq d_Y\big(h(x_1), h(x_2)\big) \leq \lambda\, d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \,\epsilon\, X \qquad (3.16)$$

Then we have

(1) If f has the bi-shadowing property with respect to the comparison class $C(X)$ and positive parameters α, β, then g has the bi-shadowing property with respect to the comparison class $C(Y)$ and positive parameters $\lambda\alpha$, β provided that $C(X)$ and $C(Y)$ are topologically conjugate by h in the sense that individual mappings in one class are topologically conjugate to a mapping in the other class by h.

(2) If g has the bi-shadowing property with respect to the comparison class $C(Y)$ and positive parameters α, $\lambda\beta$, then f has the bi-shadowing property with respect to the comparison class $C(X)$ and positive parameters $\lambda\alpha$, β provided that $C(X)$ and $C(Y)$ are topologically conjugate by h in the sense that individual mappings in one class are topologically conjugate to a mapping in the other class by h.

Proof: (1) Let $\{y_i\}_{i=1}^{\infty}$ be δ-pseudo trajectory of g with $0 \leq \delta \leq \beta$, which implies that

$$d_Y(g(y_i), y_{i+1}) < \delta,$$

and let $\varphi \in C(Y)$ satisfy

$$\delta + \sup_{y \in Y} d_Y\big(g(y), \varphi(y)\big) \leq \beta. \tag{3.17}$$

Note that condition (3.16) is equivalent to the following condition

$$d_X\big(h^{-1}(y_1), h^{-1}(y_2)\big) \leq d_Y(y_1, y_2) \leq \lambda\, d_X\big(h^{-1}(y_1), h^{-1}(y_2)\big)$$

for all $y_1, y_2 \in Y$. (3.18)

Now,

$$\begin{aligned} d_X\big(f(h^{-1}(y_i)), h^{-1}(y_{i+1})\big) &= d_X\Big(h^{-1}\big(g(y_i)\big), h^{-1}(y_{i+1})\Big) \\ &\leq d_Y(g(y_i), y_{i+1}) \\ &< \delta \end{aligned}$$

hence $x_i = h^{-1}(y_i)$ is a δ-pseudo trajectory of f. Let $x \in X$ and $\phi \in C(X)$ then using (3.16) and the conjugacy h we obtain

$$\begin{aligned} d_X\big(f(x), \phi(x)\big) &\leq d_Y\Big(h\big(f(x)\big), h\big(\phi(x)\big)\Big) \\ &= d_Y\Big(g\big(h(x)\big), \varphi\big(h(x)\big)\Big) \\ &= d_Y\big(g(y), \varphi(y)\big). \end{aligned}$$

where $y = h(x)$.

So,

$$d_X\big(f(x), \phi(x)\big) \leq d_Y\big(g(y), \varphi(y)\big) \qquad \forall x \in X, y = h(x),$$

hence

$$\sup_{x \in X} d_X\big(f(x), \phi(x)\big) \leq \sup_{y \in Y} d_Y\big(g(y), \varphi(y)\big). \tag{3.19}$$

From (3.17) and (3.19) we get for any $\phi \in C(X)$, where $\phi = h^{-1} \circ \varphi \circ h$ that

$$\delta + \sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) \le \beta.$$

But f has the bi-shadowing property. Thus there exist a true trajectory $\{w_i\}_{i=1}^{\infty}$ of ϕ such that

$$d_X(x_i, w_i) \le \alpha(\delta + \sup_{x\epsilon X} d_X(f(x), \phi(x)))$$

$$\le \alpha(\delta + \sup_{y\epsilon Y} d_Y(g(y), \varphi(y))).$$

Thus, using the second part of (3.16), we obtain

$$d_Y\big(h(x_i), h(w_i)\big) \le \lambda\, \alpha(\delta + \sup_{y\epsilon Y} d_Y(g(y), \varphi(y)))$$

hence

$$d_Y\big(y_i, h(w_i)\big) \le \lambda\, \alpha(\delta + \sup_{y\epsilon Y} d_Y(g(y), \varphi(y))).$$

Note that $h(w_i) = a_i$ is a true trajectory of φ, since

$$\varphi(a_i) = \varphi\big(h(w_i)\big) = h\big(\phi(w_i)\big) = h(w_{i+1}) = a_{i+1}.$$

This shows that g has the bi-shadowing property with respect to $C(Y)$ and with positive parameters $\lambda\alpha, \beta$.

(2) Let $\{x_i\}_{i=1}^{\infty}$ be δ-pseudo trajectory of f with $0 \le \delta \le \beta$, which implies that

$$d_X(f(x_i), x_{i+1}) < \delta,$$

and let $\phi \,\epsilon\, C(X)$ satisfy

$$\delta + \sup_{x\epsilon X} d_X\big(f(x), \phi(x)\big) \le \beta. \qquad (3.20)$$

Now,

$$d_Y\Big(g\big(h(x_i)\big), h(x_{i+1})\Big) = d_Y\Big(h\big(f(x_i)\big), h(x_{i+1})\Big)$$

$$\le \lambda\, d_X(f(x_i), x_{i+1})$$

$$\le \lambda\delta$$

hence $y_i = h(x_i)$ is a $\lambda\delta$-pseudo trajectory of g. Now Let $y \,\epsilon\, Y$ and $\varphi \,\epsilon\, C(Y)$, then using (3.18) we have

$$d_Y\big(g(y), \varphi(y)\big) \le \lambda\, d_X\Big(h^{-1}\big(g(y)\big), h^{-1}\big(\varphi(y)\big)\Big)$$

$$= \lambda\, d_X\Big(f\big(h^{-1}(y)\big), \phi\big(h^{-1}(y)\big)\Big)$$

$$= \lambda\, d_X\big(f(x), \phi(x)\big)$$

where $x = h^{-1}(y)$.

So,

$$d_Y\big(g(y), \varphi(y)\big) \le \lambda\, d_X\big(f(x), \phi(x)\big), \qquad \forall y \in Y \text{ and } x = h^{-1}(y)$$

hence

$$\sup_{y\in Y} d_Y\big(g(y), \varphi(y)\big) \le \lambda \sup_{x\in X} d_X\big(f(x), \phi(x)\big). \tag{3.21}$$

From (3.20) and (3.21) we get for any $\varphi \in C(Y)$, where $\varphi = h \circ \phi \circ h^{-1}$ that

$$\lambda\delta + \sup_{y\in Y} d_Y\big(g(y), \varphi(y)\big) \le \lambda\beta.$$

But g has the bi-shadowing property. Thus there exist a true trajectory $\{z_i\}_{i=1}^{\infty}$of φ such that

$$\begin{aligned} d_Y(y_i, z_i) &\le \alpha(\lambda\delta + \sup_{y\in Y} d_Y(g(y), \varphi(y))) \\ &\le \alpha(\lambda\delta + \lambda \sup_{x\in X} d_X(f(x), \phi(x))) \\ &= \lambda\alpha(\delta + \sup_{x\in X} d_X(f(x), \phi(x))). \end{aligned}$$

Thus, using (3.17), we obtain

$$d_X\big(h^{-1}(y_i), h^{-1}(z_i)\big) \le \lambda\alpha(\delta + \sup_{x\in X} d_X(f(x), \phi(x)))$$

hence

$$d_X\big(x_i, h^{-1}(z_i)\big) \le \lambda\alpha(\delta + \sup_{x\in X} d_X(f(x), \phi(x))).$$

Note that $h^{-1}(z_i) = a_i$ is a true trajectory of ϕ, since

$$\phi(a_i) = \phi\big(h^{-1}(z_i)\big) = h^{-1}\big(\varphi(z_i)\big) = h^{-1}(z_{i+1}) = a_{i+1}.$$

This shows that f has the bi-shadowing property with respect to $C(X)$ and with positive parameters $\lambda\alpha$ and β.

Thus, the proof of the theorem is complete. □

Remark 3.2.1: In the previous theorem if $\lambda = 1$, then we conclude that the map $f: X \to X$ has the bi-shadowing property with respect to positive parameters α and β iff $g: Y \to Y$ has the bi-shadowing property with respect to the same parameters.

CHAPTER FOUR

Conclusions and Recommendations

In this work, we studied the asymptotic behavior of discrete dynamical system generated by a continuous mappings on a metric space. We established some of results regarding the concept of shadowing property and generalized these results in the context of bi-shadowing property. We discuss the relationship of bi-shadowing property between the product system and its subsystems. Also, we introduce a new definition of a bi-shadowing property for a pair of systems, which we call it mutually bi-shadowing. In the end it was shown that the bi-shadowing property is invariant under topological conjugacy with certain conditions.

Based in our work, the results that we have obtained can be extended to a nonautonomous case of dynamical system, we may deal with this result in the future.

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